PHYSICAL REVIEW E VOLUME 57, NUMBER 4 APRIL 1998

Effect of boundary conditions on fluctuations measures

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(Received 28 July 1997; revised manuscript received 4 December 1997)

A change in boundary conditions (BC) from uniform Dirichlet to nonidentical BC on the edges of a triangular billiard often brings about a dramatic change in quantum spectral fluctuations. We provide a theory for this based on periodic orbits and show that nonidentical BC on adjacent edges can lead to a *quantum splitting* of periodic orbit families, which results in a significant change in the form factor. Thus, the classical spectrum alone cannot determine quantum correlations.
[S1063-651X(98)50304-2]

PACS number(s): 05.45.+b, 03.65.Sq

Recent studies [1,2] on fluctuation measures in chaotic systems seem to indicate that the quantum correlations are fully determined by the classical spectrum of the Perron-Frobenius operator [3]. Using different approaches, Agam, Altshuler, and Andreev (AAA) [1] and Bogomolny and Keating (BK) [2] show that the diagonal and off-diagonal parts of the density-density correlation are related to a purely classical quantity, which under some approximation reduces to the classical ζ function [3].

There are several fallouts of the AAA-BK theory. One that has been scrutinized recently by Prange [4] concerns possible deviations from random matrix theory results and the conditions under which this can be observed. Another consequence (and one that is of relevance here) is the absence of parity effects in fluctuation measures. In other words, the AAA-BK theory predicts that quantum systems having the same classical dynamics exhibit identical quantum correlations. There is, however, a tacit assumption in the BK approach which leads us to this conclusion: that degenerate periodic orbits with identical actions and stabilities also have the same quantum phase. There can be instances, however, when this is not true. For example, arithmetic billiards abound in degenerate periodic orbits and exhibit Poisson fluctuations when the boundary conditions are Dirichlet [5]. However, when the boundary conditions are not identically Dirichlet, pairs of degenerate periodic orbits can have phases differing by π, leading to a net decrease in the form factor [6].

The effect of boundary conditions can be even more significant in planar triangular billiards, and we shall deal with these henceforth. Of these, the ones that are integrable have internal angles of the form π/n_i and in all these cases, degeneracies exist in the classical periodic orbit actions of topologically distinct orbits leading to nonuniversal spectral fluctuations [7,8]. Thus, the spectral rigidity [9], $\Delta_3(L)$, increases with a slope larger than $\frac{1}{15}$ [10] in the region $L \ll L_{\max}$, where $L_{\max}$ is determined by the frequency of slowest oscillation in the quantum density $\rho(E)$ [11].

Generic rational triangles, on the other hand, have internal angles of the form $\pi m_i/n_i$, where $\Pi_{i=1}^{3} m_i \neq 1$. These are referred to as pseudointegrable (PI) billiards [12–14]. As in integrable systems, their invariant surface in phase space is two-dimensional but the topology is that of a sphere with multiple holes and not a torus [15]. This difference leads to a rather dramatic change in the quantum eigenstates. The eigenfunctions, for example, often exhibit irregular nodal patterns and a Gaussian amplitude distribution [16], while fluctuation measures display a behavior [7,17], that ranges from the integrable [7] to the chaotic [18–20,7] limits.

There are several interesting questions concerning quantum fluctuations that polygonal billiards throw up. A point that has often been debated is the role of diffractive periodic orbits in determining spectral measures [21]. Admittedly, the quantum spectrum does know about these orbits [22], though its importance in determining spectral measures is possibly negligible [23]. A related question concerns the effect of boundary conditions on spectral fluctuations. To illustrate this, we refer to Fig. 1 where the rigidity, Δ_3, is plotted as a function of L for the right triangle ($\pi/2$, $\pi/3$, $\pi/6$) with (a) Dirichlet boundary condition on all three edges and (b)

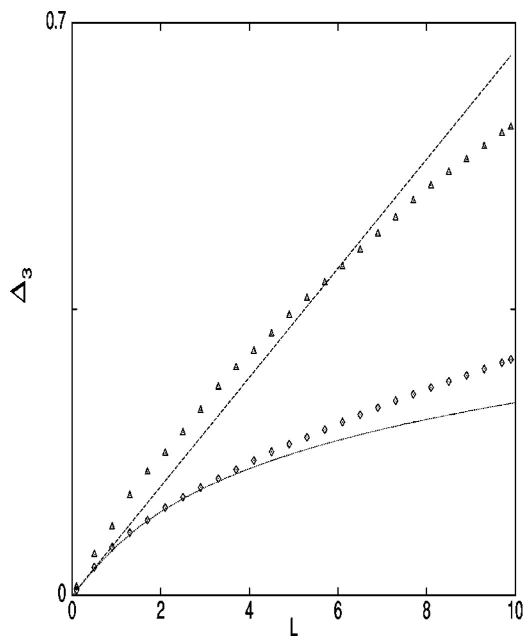

FIG. 1. $\Delta_3(L)$ for the ($\pi/2$, $\pi/3$, $\pi/6$) triangle with (a) Dirichlet boundary conditions on all edges (△) and (b) Neumann boundary conditions on the edges enclosing the right angle and Dirichlet on the third (◇). The averaging interval is $[\epsilon_n - \Delta\epsilon,\ \epsilon_n + \Delta\epsilon]$ with $\epsilon_n = 800$ and $\Delta\epsilon = 300$. The straight line and the smooth curve are, respectively, the Poisson ($L/15$) and GOE results.

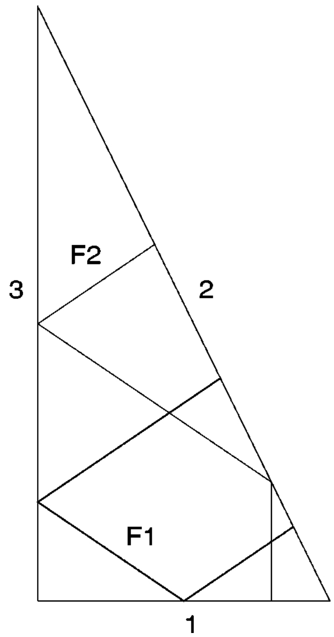

FIG. 2. Two periodic orbits $F1$ (thick line) and $F2$ belonging to the same classical family.

Neumann boundary condition on the edges enclosing the right angle and Dirichlet BC on the third. One might argue that the crossover is related to the fact that case (a) is integrable while (b) is not [24]. We have thus verified that there is indeed a shift with boundary conditions in genuine pseudo-integrable enclosures such as the $(\pi/2,3\pi/10,\pi/5)$ triangle.

At first glance, it may seem that apart from an overall phase factor, the contribution of each periodic orbit family is identical for the two cases of the $(\pi/2,\ \pi/3,\ \pi/6)$ triangle. We shall demonstrate here that this is not the case. To this end, consider the family of periodic orbits shown in Fig. 2. Orbits $F1$ and $F2$ belong to the same family *classically* though the quantum phase accumulated by these differ by π when edges 1 and 3 have Neumann boundary conditions [we refer to this as case (b), while case (a) denotes Dirichlet BC on all edges]. In other words, the family of periodic orbits split up in the *quantum-mechanical* sense in case (b), while case (a) preserves the full classical family. The semiclassical density of states,

$$\rho(E)\simeq\rho_{av}(E)+\sqrt{\frac{1}{8\pi^3}}\sum_p\sum_{r=1}^{\infty}\frac{a_p}{\sqrt{krl_p}}\times\cos\left(krl_p-\frac{\pi}{4}-rn_p\pi\right) \tag{1}$$

for the two cases are thus distinct. In the above, $E=k^2$ while a_p refers to the area occupied by a primitive periodic orbit family characterized by their length l_p and *the number, n_p of reflections from Dirichlet edges*. Equation (1) has only the leading order fluctuation in the density of states. It neglects the contribution of isolated periodic orbits and diffractive periodic orbits. Also note that orbits that occur in families necessarily undergo an even number of reflections from the edges so that the net phase in case (a) is zero.

With this background, we are now ready to explore the effect of BC on spectral measures. For two-point correlations such as $\Delta_3(L)$ and $\Sigma_2(L)$, the central object is the form factor $\phi(T)=\int_{-\infty}^{\infty}R_2(x)\exp(ixT/\hbar)dx$, where $R_2(x)$ is the two-point spectral correlation function $[R_2(x)=\langle\rho(E+x)\rho(E)\rangle]$. Expressed in terms of periodic orbits, $\phi(T)=\langle\Sigma_i\Sigma_jA_iA_j\cos(S_i-S_j)\delta(T-(T_i+T_j)/2)\rangle$ where $A_i=Ca_i/\sqrt{kl_i}$ for marginally unstable billiards, $S_i=kl_i-\pi/4-n_i\pi$, $T_i=\partial S_i/\partial E$ and $C=\sqrt{1/(32\pi^3)}$.

It is customary to analyze the diagonal and off-diagonal parts of $\phi(T)$ separately and we first show that for case (b), the diagonal contribution $\phi_D(T)=\langle\Sigma_iA_i^2\delta(T-T_i)\rangle$ is smaller as compared to case (a). Let us assume that the family labeled by i splits up quantum mechanically in case (b) into two parts, occupying areas a_{i1} and a_{i2} respectively where $a_{i1}+a_{i2}=a_i$. Its contribution to $\phi_D(T)$ is thus proportional to $a_{i1}^2+a_{i2}^2$, while in case (a), it is proportional to $a_{i1}^2+a_{i2}^2+2a_{i1}a_{i2}$. Further, since the two parts of the classical family have a different phase in case (b), there is an off-diagonal (OD) contribution from within this classical family. Its magnitude is proportional to $2a_{i1}a_{i2}\cos(\pi)$ so that the net decrease in contribution of a single classical family is proportional to $4a_{i1}a_{i2}$.

Note that the off-diagonal part of the form factor has cross contributions as well where parts of two distinct classical families are involved. When the classical dynamics is integrable and no QS occurs, the OD contributions average to zero in the absence of *degeneracies* amongst periodic orbit actions. Thus, the diagonal contribution equals the asymptotic value of $\phi(T)$, which equals $\rho_{av}/2\pi$. This asymptotic law is referred to as the semiclassical sum rule [11]. Even in the presence of QS, the semiclassical sum rule holds. Thus, cross terms involving parts of distinct classical families do contribute in case (b). In summary then, the following comparison between cases (a) and (b) can be made when there are *no degeneracies* in the lengths of topologically distinct periodic orbits. For $T\ll T_H$, the form factor equals

$$\phi(T)=\left\langle C^2\sum_i\frac{a_i^2}{kl_i}\delta(T-T_i)\right\rangle \quad \text{case (a)}, \tag{2}$$

$$=\left\langle C^2\sum_i\frac{\{a_i(2\alpha_i-1)\}^2}{kl_i}\delta(T-T_i)\right\rangle \quad \text{case (b)}, \tag{3}$$

while in both cases, $\phi(T)=\rho_{av}(E)/(2\pi)$ as $T\to\infty$. Here $a_{i1}=\alpha_ia_i$, $a_{i2}=(1-\alpha_i)a_i$ and T_H is the Heisenberg time.

Note that in an integrable enclosure, the areas a_i are identical for almost all orbit families so that for case (a), a straightforward application of the proliferation law for periodic orbit families leads to the conclusion that $\phi(T)$ is constant and equals $\rho_{av}(E)/2\pi$. For case (b), however, the factor $(2\alpha_i-1)^2$ varies with the orbit and depending on the splitting mechanism, $\phi(T)$ may be explicitly T dependent even in an ''integrable'' enclosure [25].

In order to concretize these notions, let us take another look at the $(\pi/2,\pi/3,\pi/6)$ enclosure of Fig. 1. For this integrable billiard, the length spectrum can be expressed in terms

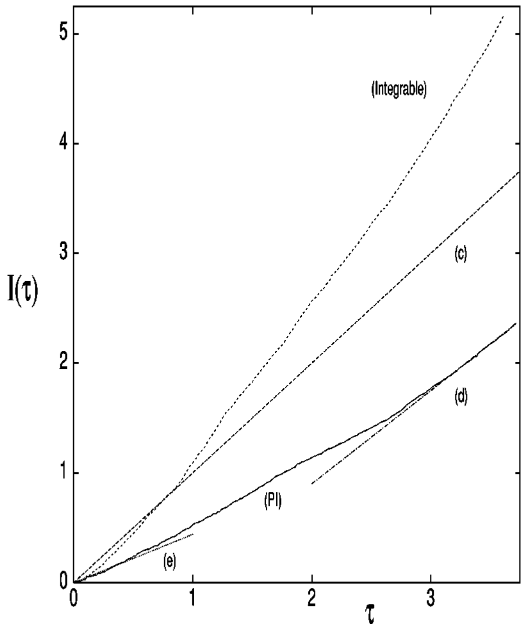

FIG. 3. Plot of $I(\tau)$ for the ($\pi/2$, $\pi/3$, $\pi/6$) billiard. The curve marked (integrable) is for case (a), while (PI) represents case (b). Also shown are three lines with slopes 1.0, 0.85, and 0.45 marked (c), (d), and (e), respectively. For averaging, see Fig. 1.

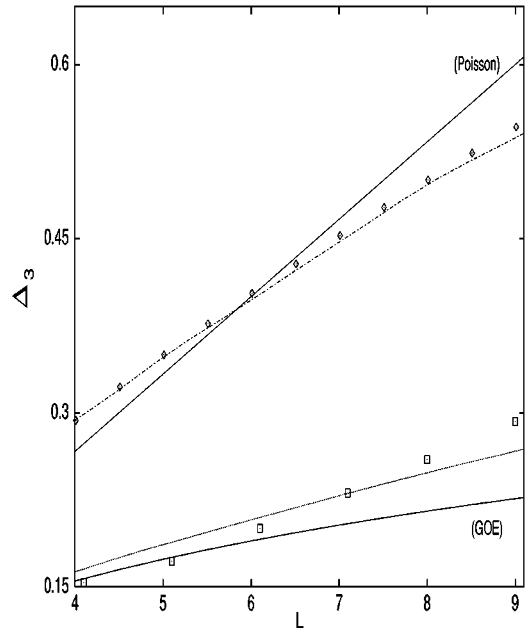

FIG. 4. The chain and dotted curves are the exact values of $\Delta_3(L)$ for case (a) and (b), respectively. The diamonds and squares are estimates obtained using periodic orbits for case (a) and (b), respectively. For averaging, see Fig. 1.

of winding numbers on tori and it is easy to verify that there exists degeneracies in the lengths of topologically distinct periodic orbits. For case (a) then, the sum in Eq. (2) is over distinct lengths l_i instead of topologically distinct orbits. Correspondingly, the area a_i should now be interpreted as the total area occupied by all orbit families having length l_i. An immediate consequence is that $\phi(T)$ is no longer a constant for all T since the degree of degeneracy varies with length [8]. A plot of $I(\tau)=2\pi/\rho_{av}\int\phi(\tau')d\tau'$ with respect to $\tau=T/(2\pi\rho_{av})$ is provided in Fig. 3. For generic integrable systems without degeneracies in periodic orbit lengths, $I(\tau)=\tau$, while in the present case, one observes a nonlinear increase.

For case (b), the splitting mechanism needs to be incorporated and for this example, the ratio in which certain orbit families split up has been arrived at by Shudo [26]. As before, the sum in Eq. (3) now refers to distinct lengths while the effective area $a_i(2\alpha_i-1)$ (denoted by $\overline{a_i}$) is the sum of all areas occupied by degenerate orbit families *weighted appropriately by the phases*. Thus $\overline{a_i}=\Sigma_k(-1)^{n_k}a_k$, where a_k is the area occupied by a family having length l_i and which undergoes n_k reflections from the Dirichlet edge.

Once more, rather than the asymptotic proliferation rate of periodic orbit families, it is the variation of $\overline{a_i}$ with length [27] which determines the form factor. A plot of $I(\tau)$ for case (b) (see Fig. 3) reveals a nonlinear increase having a smaller overall slope and a form that is distinct from case (a). Thus, a change in BC from uniform Dirichlet to nonuniform BC leads to a significant change in the form factor. This difference indeed shows up in the spectral rigidity $\Delta_3(L)$. In Fig. 4, we compare the predictions of periodic orbit theory with the exact (numerical) values of Δ_3 in the range $4\leqslant L\leqslant 10$ [28] for cases (a) and (b). While the agreement for case (a) is excellent, the predictions of periodic orbit theory capture the overall behavior in case (b).

The discussion so far holds for all polygonal billiards where adjacent edges enclosing an angle of the form π/n_i have nonidentical (NI) boundary conditions. In such cases, periodic orbit families do not split up at this angle *classically*, though as demonstrated earlier, they can split up *semiclassically*. For angles of the form $m_i\pi/n_i$ ($m_i>1$), however, orbit families do split up classically and traverse different paths, thereby reducing the extent of periodic orbit families. Thus, different sets of boundary conditions only result in an overall phase factor for each family and hence do

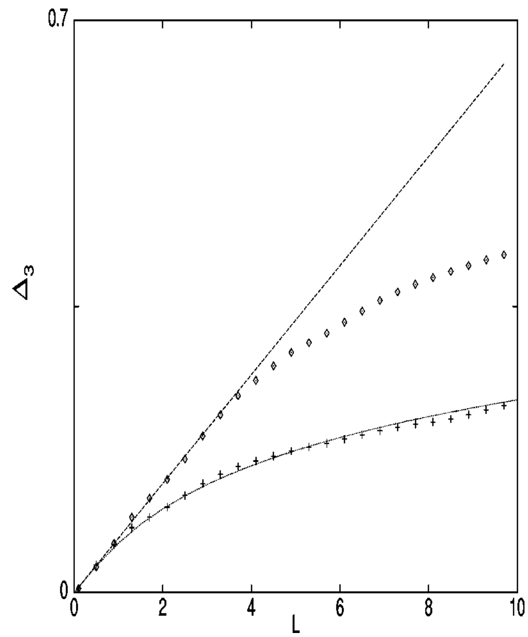

FIG. 5. Rigidity for the irrational triangle ($\pi/2$, $\pi/\sqrt{9.1}$). Case (a) exhibits Poisson fluctuations ($\diamond$), while case (b) shows GOE fluctuations (+) when $\epsilon_n=500$ and $\Delta\epsilon=150$.

not significantly affect spectral measures. In exceptional cases, however, the effect of boundary conditions can be significant. This can be observed when the angle is of the form $m_i\pi/n_i$ ($m_i>1$) but close to an *integrable* wedge. As an example, consider the rigidity $\Delta_3(L)$ for the irrational triangle ($\pi/2$, $\pi/\sqrt{9.1}$), which is close to the integrable ($\pi/2$, $\pi/3$) enclosure (Fig. 5). Case (a) clearly exhibits fluctuations close to Poisson, while case (b) shows typical GOE fluctuations for the energy range considered. Note that the triangle has infinite genus though over short time scales (less than the Heisenberg time) the dynamics hovers around its integrable counterpart, while even after 10^9 reflections from the boundary, parts of the constant energy surface remain unexplored. The two *nonintegrable* acute angles, however, serve to split up periodic orbit families though the lengths of the resulting families remain close to that of the original family in the integrable enclosure. This subtle reorganization of periodic orbit families leads to Poisson fluctuations in case (a), since the degeneracies in orbit actions which exist for the ($\pi/2$, $\pi/3$) triangle get lifted in the ($\pi/2$, $\pi/\sqrt{9.1}$) enclosure. On the other hand, when the lifting of degeneracies is accompanied by a difference in phase between two *split* families [case (b)], the change is significant and leads to GOE-like fluctuations for the energy range considered [29].

In summary, we have demonstrated that a change from uniform Dirichlet to nonidentical boundary conditions on the edges of triangular billiard can lead to significant changes in fluctuation measures. This can be observed when the angle enclosed by the edges with NI boundary conditions is of the form π/n or sufficiently close to it [30]. The mechanism involved is *quantum splitting* due to which adjacent families having (almost) identical lengths acquire different phases, leading to a significant drop in contribution from both the diagonal and off-diagonal terms in the form factor. In particular, we have shown that it is possible to explain the spectral fluctuations of the ($\pi/2$, $\pi/3$, $\pi/6$) triangle when the boundary conditions are not identically Dirichlet using periodic orbit theory. We conclude by noting that there exist quantum systems whose density correlations cannot be determined fully by the classical spectrum.

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